DENSITY AND EQUIDISTRIBUTION OF ONE-SIDED HOROCYCLES OF A GEOMETRICALLY FINITE HYPERBOLIC SURFACE

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ABSTRACT. On geometrically finite negatively curved surfaces, we give necessary and sufficient conditions for a one-sided horocycle $(h^s u)_{s\geq 0}$ to be dense in the nonwandering set of the geodesic flow. We prove that all dense one-sided orbits $(h^s u)_{s\geq 0}$ are equidistributed, extending results of [**Bu**] and [**Scha2**] where symmetric horocycles $(h^s u)_{-R\leq s\leq R}$ were considered.

1. INTRODUCTION

Hedlund [**H**] proved that the horocyclic flow $(h^s)_{s\in\mathbb{R}}$ on the unit tangent bundle of a finite volume hyperbolic surface is minimal, that is all nonperiodic orbits $(h^s v)_{s\geq 0}$ (called in [**H**] "right-semihorocycles", and here positive half-horocycles) are dense.

On *geometrically finite surfaces*, i.e. surfaces whose fundamental group is finitely generated, it is known (see [**E**], [**Da**]) that all nonwandering and non periodic orbits of the horocyclic flow are *dense* in the sense that the closure of $(h^s v)_{s\in\mathbb{R}}$ contains the nonwandering set of the geodesic flow. On general hyperbolic surfaces, keeping this definition of "dense" horocyclic orbit, we know ([**E**] [**Da**] [**St**]) how to characterize dense orbits: a horocycle is dense iff it is centered at a horospherical point.

However, as soon as the fundamental group of the surface is of the second kind, i.e. its limit set is strictly included in the boundary at infinity S^1 (see section 2), we can easily find horocycles $(h^s u)_{s\in\mathbb{R}}$ that are globally dense in the nonwandering set Ω of the geodesic flow (in the sense that $\overline{(h^s u)_{s\in\mathbb{R}}} \supset \Omega$), but with one side dense and the other not.

In this note, answering a question of O. Sarig, we characterize these horocycles with one side dense and the other not. If $u \in T^1S$, and $\tilde{u}$ is any of its lifts on the unit tangent bundle $T^1\mathbb{D}$ of the hyperbolic disc, we denote by $u^- \in S^1$ (resp. u^+) the negative (resp. positive) endpoint in the boundary $S^1 = \partial\mathbb{D}$ of the geodesic line defined by $\tilde{u}$. We prove:

Theorem 1.1. *Let S be a geometrically finite hyperbolic surface. Let $u \in T^1S$ be s.t. its full unstable horocyclic orbit $(h^s u)_{s\in\mathbb{R}}$ is dense in Ω. Then the positive half-horocycle $(h^s u)_{s\geq 0}$ is dense in Ω iff u^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$ (where the circle S^1 is oriented in the counterclockwise direction).*

On general hyperbolic surfaces, this theorem remains valid for vectors $u \in T^1S$ that are periodic for the geodesic flow (see proposition 3.12).

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In fact, inspsired by ideas of [**C**], we introduce the notion of *right horospherical vector*, and prove (proposition 3.9) that on general hyperbolic surfaces, a positive half-horocycle $(h^s u)_{s\geq 0}$ is dense iff u is a right horospherical vector. We deduce Theorem 1.1 from the fact that on geometrically finite surfaces, right horospherical vectors are easy to characterize.

Our initial motivation was the study of equidistribution properties of horocycles. Furstenberg's unique ergodicity result [**F**] for the horocyclic flow ensures that on the unit tangent bundle T^1S of a compact hyperbolic surface, all horocyclic orbits are equidistributed towards the unique (h^s)-invariant measure λ: for all $u\in T^1S$ and $f:T^1S\to\mathbb{R}$ continuous, $\displaystyle\frac{1}{T}\int_0^T f\circ h^s u\,ds\to\int_{T^1S} f\,d\lambda$, where λ is the Liouville measure. Of course, the same result holds for $(h^s u)_{s\leq 0}$.

This result was extended by Dani and Smillie [**DS**] to finite volume hyperbolic surfaces: all nonperiodic one-sided orbits $(h^s u)_{s\geq 0}$ are equidistributed towards λ.

On geometrically finite hyperbolic surfaces, there is [**Ro**] [**Bu**] a unique (h^s)-invariant ergodic measure m that has full support in the nonwandering set of (h^s); and it is infinite. Therefore, as in Hopf ergodic theorem, one considers ratios $\displaystyle\frac{\int_{-T}^T f\circ h^s u\,ds}{\int_{-T}^T g\circ h^s u\,ds}$ and one can prove [**Bu**][**Scha2**] that they converge to $\displaystyle\frac{\int_{T^1S} f\,dm}{\int_{T^1S} g\,dm}$ for all continuous functions $f,g:T^1S\to\mathbb{R}$ with compact support, and all nonwandering and non periodic vectors $u\in T^1S$. In these two articles, equidistribution is obtained for symmetric horocycles $(h^s u)_{-T\leq s\leq T}$ only, and not for one-sided horocycles $(h^s u)_{0\leq s\leq T}$. Symmetric averages are very natural from a geometric point of view, but not from the ergodic point of view, where a difference of behaviour between the negative and the positive orbit is an interesting phenomenon.

In theorem 1.1, we characterized dense horocycles that have one side dense and the other not. For these horocycles, one cannot hope equidistribution of both one-sided orbits. However, according to Hopf ergodic theorem, almost all one-sided horocycles should be equidistributed towards m.

On geometrically finite hyperbolic surfaces, the above phenomenon of dense horocycles with a nondense side is the only obstruction to the equidistribution of one-sided horocycles. Indeed, with methods of [**Scha1**] and [**Scha2**], we get:

Theorem 1.2. *Let S be a geometrically finite surface, and $u\in T^1S$ such that $(h^s u)_{s\geq 0}$ is dense in the nonwandering set of the geodesic flow. Then $(h^s u)_{s\geq 0}$ is equidistributed towards the unique invariant measure m which has full support in the nonwandering set of $(h^s)_{s\in\mathbb{R}}$.*

In other words, for all continuous functions with compact support $f,g:T^1S\to\mathbb{R}$, with $\int_{T^1S} g\,dm>0$, we have

$$\frac{\int_0^T f\circ h^s u\,ds}{\int_0^T g\circ h^s u\,ds}\to\frac{\int_{T^1S} f\,dm}{\int_{T^1S} g\,dm}\,,\quad \text{when}\quad T\to+\infty\,.$$

Note that periodic orbits are obviously equidistributed to the Lebesgue measure on the orbit. Of course, theorem 1.2 also holds for negative orbits $(h^s u)_{s\leq 0}$.

Most results extend to surfaces of variable negative curvature. However, to avoid too many preliminaries, we postpone the discussion on such surfaces to the end of the paper.

Section 2 is devoted to preliminaries. Theorem 1.1 is proved in section 3, where we also discuss the case of geometrically infinite surfaces, and theorem 1.2 in section 4.

I thank warmly O. Sarig for the question at the origine of this note, F. Dal'bo for her comments on the first version of this work, Yves Coudene for several helpful discussions, and all members of the former french ANR project on infinite ergodic theory for our fruitful work together.

2. Preliminaries

Hyperbolic geometry. The hyperbolic disc $\mathbb{D} = D(0,1)$ is endowed with the metric $\frac{1}{4}\frac{dx^2}{(1-|x|^2)^2}$. Let o be the origin of the disc. Denote by $\pi : T^1\mathbb{D} \to \mathbb{D}$ the canonical projection. The *boundary at infinity* is $S^1 = \partial\mathbb{D}$.

The map $z \in \mathbb{D} \mapsto \frac{i(1+z)}{1-z}$ is an isometry between $\mathbb{D}$ with the above metric and the upper half plane $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$ endowed with the hyperbolic metric $\frac{dx^2+dy^2}{y}$. Therefore, the group of isometries preserving orientation of $\mathbb{D}$ identifies with $PSL(2,\mathbb{R})$ acting by homographies on $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$. It acts simply transitively on the unit tangent bundle $T^1\mathbb{D}$, so that we identify these two spaces through the map which sends the unit vector $(1,0)$ tangent to $\mathbb{D}$ at $o = (0,0)$ on the identity element of $PSL(2,\mathbb{R})$.

The *Busemann cocycle* is the continuous map defined on $S^1 \times \mathbb{D}^2$ by

$$\beta_\xi(x,y) := \lim_{z\to\xi} \left(d(x,z) - d(y,z)\right) .$$

Define the map $v \in T^1\mathbb{D} \mapsto (v^-, v^+, \beta_{v^-}(\pi(v), o))$, where $v^\pm$ are the endpoints in S^1 of the geodesic defined by v, and $\pi(v) \in \mathbb{D}$ is the basepoint in S of v. It defines a homeomorphism between $T^1\mathbb{D}$ and $\partial^2\mathbb{D} \times \mathbb{R} := S^1 \times S^1 \setminus \text{Diagonal} \times \mathbb{R}$.

Let Γ be a discrete subgroup of $PSL(2,\mathbb{R})$. Its *limit set* is $\Lambda_\Gamma = \overline{\Gamma o} \setminus \Gamma o \subset S^1$. The group Γ acts properly discontinuously on the *ordinary set* $S^1 \setminus \Lambda_\Gamma$, which is a countable union of intervals.

A point $\xi \in \Lambda_\Gamma$ is a *radial* limit point if it is the limit of a sequence $(\gamma_n.o)$ of points of $\Gamma.o$ that stay at bounded distance of the geodesic ray $[o\xi)$ joining o to ξ. Let Λ_{rad} denote the *radial limit set*.

The point $\xi \in \Lambda_\Gamma$ is *horospherical* if any horoball centered at ξ contains infinitely many points of $\Gamma.o$. In particular, Λ_{rad} is included in the horospherical set Λ_{hor}.

An isometry of $PSL(2,\mathbb{R})$ is *hyperbolic* if it fixes exactly two points of S^1, it is *parabolic* if it fixes exactly one point of S^1, and *elliptic* in the other cases. Let $\Lambda_{\mathrm{p}} \subset \Lambda_\Gamma$ denote the set of *parabolic* limit points, that is the points of Λ_Γ fixed by a parabolic isometry of Γ.

Any hyperbolic surface is the quotient $S = \Gamma\backslash\mathbb{D}$ of $\mathbb{D}$ by a discrete subgroup Γ of $PSL(2,\mathbb{R})$ without elliptic element. Its unit tangent bundle T^1S identifies with $\Gamma\backslash PSL(2,\mathbb{R})$.

In this note, we always assume Γ be *nonelementary*, that is $\#\Lambda_\Gamma = +\infty$. Moreover, we are mainly interested in *geometrically finite surfaces* S, i.e. surfaces whose fundamental group Γ is finitely generated. In such cases, the limit set Λ_Γ is the disjoint union of Λ_{rad} and Λ_{p} [**Bow**]. Moreover, the surface is a disjoint union of a compact part C_0, finitely many cusps (isometric to $\{z \in \mathbb{H},\ Im\, z \geq cst\}/\{z \mapsto z+1\}$), and finitely many 'funnels' (isometric to $\{z \in \mathbb{H},\ Re(z) \geq 0,\ 1 \leq |z| \leq a\}/\{z \mapsto az\} = \{z \in \mathbb{H},\ Re(z) \geq 0\}/\{z \mapsto az\}$, for some $a > 1$.

When S is compact, $\Lambda_\Gamma = \Lambda_{\rm rad} = S^1$. It is said *convex-cocompact* when it is a geometrically finite surface without cusps. In this case, $\Lambda_\Gamma = \Lambda_{rad}$ is strictly included in S^1 and Γ acts cocompactly on the set $(\Lambda_\Gamma \times \Lambda_\Gamma) \setminus \text{Diagonal} \times \mathbb{R} \subset T^1\mathbb{D}$. (We identify now the two homeomorphic spaces $T^1\mathbb{D}$ and $S^1 \times S^1 \setminus \text{Diagonal} \times \mathbb{R}$.) When S has finite volume, there are no funnels and $\Lambda_\Gamma = S^1$.

Geodesic and horocycle flows. A hyperbolic geodesic in $\mathbb{D}$ is a diameter or a half-circle orthogonal to S^1. A *horocycle* of $\mathbb{D}$ is a circle tangent to S^1. It can also be defined as a level set of a Busemann function. A *horoball* is the (euclidean) disc delimited by a horocycle. A vector $v \in T^1\mathbb{D}$ is tangent to a unique geodesic of $\mathbb{D}$. Moreover, it is orthogonal to exactly two horocycles passing through its basepoint $\pi(v)$, and tangent to S^1 respectively at v^+ and v^-. The set of vectors $w \in T^1\mathbb{D}$ such that $w^- = v^-$ and based on the same horocycle tangent to S^1 at v^- is the *strong unstable horocycle* or strong unstable manifold $W^{su}(v) \subset T^1\mathbb{D}$ of v The *strong stable manifold* $W^{ss}(v)$ is defined in the same way.

The *geodesic flow* $(g^t)_{t\in\mathbb{R}}$ acts on $T^1\mathbb{D}$ by moving a vector v of a distance t along its geodesic. In the identification of $T^1\mathbb{D}$ with $PSL(2,\mathbb{R})$, this flow corresponds to the right action by the one-parameter subgroup

$$\left\{a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \, t \in \mathbb{R}\right\}.$$

The *strong unstable horocyclic flow* $(h^s)_{s\in\mathbb{R}}$ acts on $T^1\mathbb{D}$ by moving a vector v of a distance $|s|$ along its strong unstable horocycle. There are two possible orientations for this flow, and we consider the choice corresponding to the right action by the one parameter subgroup

$$\left\{n_s := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \, s \in \mathbb{R}\right\}$$

on $PSL(2,\mathbb{R})$. This flow turns vectors along their strong unstable horocycle, so that $\{h^s v, \, s \in \mathbb{R}\} = W^{su}(v)$. The horocyclic orbits are the *strong unstable manifolds* of the geodesic flow in the sense that

$$W^{su}(v) = \{w \in T^1\mathbb{D}, \, d(g^{-t}v, g^{-t}w) \to 0 \text{ quand } t \to +\infty\}\,.$$

Moreover, it satisfies

$$g^t \circ h^s = h^{se^t} \circ g^t\,.$$

These two right-actions are well defined on the quotient space $T^1S \simeq \Gamma\backslash PSL(2,\mathbb{R})$. The nonwandering set Ω of the geodesic flow is the set $\Gamma\backslash(\Lambda_\Gamma^2 \times \mathbb{R})$. The horocyclic flow is topologically transitive (see [**Da**]) in the sense that there exists $u \in T^1S$ such that $\overline{(h^s u)_{s\in\mathbb{R}}} \supset \Omega$. It allows to see that the nonwandering set $\mathcal{E}$ of the horocyclic flow is the set $\Gamma\backslash(\Lambda_\Gamma \times S^1 \times \mathbb{R})$ of vectors such that $v^- \in \Lambda_\Gamma$.

In our situation (nonelementary hyperbolic surfaces) we know that the length spectrum of the fundamental group Γ of S is nonarithmetic, that is the set $\{l(\gamma)\}$ of lengths of closed geodesics generates a dense subgroup of $\mathbb{R}$. We will use this crucial fact in the sequel.

Local product structure of the geodesic flow. The geodesic flow on the unit tangent bundle of any hyperbolic surface (including $\mathbb{D}$) is a *hyperbolic flow*. In particular, it has a (uniform) *local product structure*: *for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $d(u,v) \le \delta$, there is a vector $w = [u,v]$ in $W^{su}_\varepsilon(g^t u) \cap W^{ss}_\varepsilon(v)$, where*

$W^{ss}_\varepsilon(v)$ is the intersection of the strong stable horocycle of v with the ball centered at v of radius ε and $|t| \le \varepsilon$.

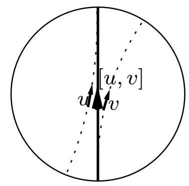

FIGURE 1. Local product in the hyperbolic disc $\mathbb{D}$

3. Density of positive half-horocycles

Recall that $W^{su}(v) = \{h^s v,\, s \in \mathbb{R}\}$ is compact iff $v^- \in \Lambda_p$, and dense in Ω iff $v^- \in \Lambda_{hor}$ (see [**Da**]). Denote by $W^{su}_+(v) = \{h^s v,\, s \ge 0\}$ the positive half-horocycle.

We suppose in the sequel that S^1 is oriented in the counterclockwise direction.

Geometry of funnels.

Remark 3.1. If the surface $S = \mathbb{D}/\Gamma$ has a funnel isometric to $\{z \in \mathbb{H},\, Re(z) \ge 0\}/\{z \mapsto az\}$, with $a > 1$, the geodesic line $Re(z) = 0$ of $\mathbb{D}$ induces on the quotient the closed geodesic closing the funnel. Any geodesic line crossing this closed geodesic and entering into the funnel never returns back to the other side. In particular, the limit set Λ_Γ does not intersect the right half line $\mathbb{R}^*_+$.

From this elementary remark, we deduce the following key facts.

Fact 3.2. On a geometrically finite hyperbolic manifold, the only points on the boundary of an interval of $S^1 \setminus \Lambda_\Gamma$ are hyperbolic. More precisely, both extremities of such an interval are the endpoints $p^\pm$ of the axis of a lift of the closed geodesic bording the corresponding funnel.

Fact 3.3. Assume S be geometrically finite. If $v^- \in \Lambda_{hor}$ is the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, then $W^{su}_+(v)$ is not dense in Ω and $(g^{-t}v)_{t\ge 0}$ is asymptotic to the closed geodesic turning around a funnel.

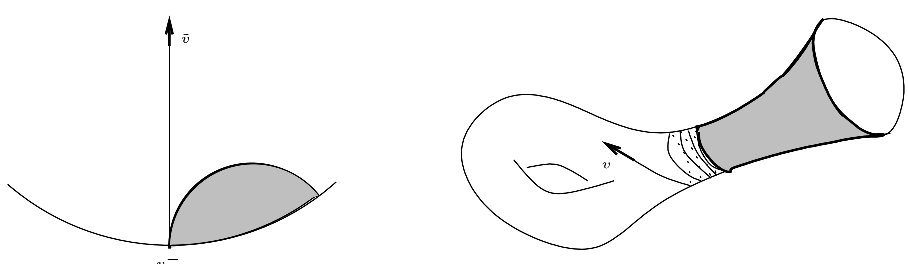

FIGURE 2. A vector whose right horocycle is not dense in Ω

Right horocyclic vectors and right horocyclic points. If $v \in T^1\mathbb{D}$, we denote by $Hor(v) \subset \mathbb{D}$ the horoball centered at v^- and containing the base point of v in its boundary. We denote by $Hor^+(v) \subset Hor(v)$ the "right part" of the horoball, i.e. the set of basepoints of vectors of $\cup_{t\geq 0} W^{su}_+(g^{-t}v)$.

Fix a point $o \in \mathbb{D}$. If S is a geometrically finite surface, we assume that o belongs to a lift of the compact part of S.

In [**C**], a vector $v \in T^1 S$ is called *horospherical* if there exists $z \in \Omega$, $t_i \to +\infty$ and $v_i \in W^{su}(v) \cap \Omega$ s.t. $g^{-t_i} v_i \to z \in \Omega$. It is equivalent to saying that $v^- \in \Lambda_{hor}$, that is that all horoballs centered at v^- contain infinitely many points of the orbit $\Gamma.o$ (see lemma 3.7 below for a proof).

Definition 3.4. If $v \in T^1\mathbb{D}$, and $\alpha > 0$, we define the *cone* of width α around v as the set $\mathcal{C}(v, \alpha)$ of points at distance at most α from the geodesic ray $(g^{-t}v)_{t\geq 0}$ inside the horoball $Hor(v)$.

Definition 3.5. Let S be a nonelementary hyperbolic surface. A vector $v \in T^1 S$ is a *right horocyclic vector* if for a lift $\tilde{v} \in T^1 S$, for all $\alpha > 0$ and $D > 0$, the orbit $\Gamma.o$ intersects the right horoball $Hor^+(g^{-D}\tilde{v})$ minus the cone $C(g^{-D}\tilde{v}, \alpha)$.

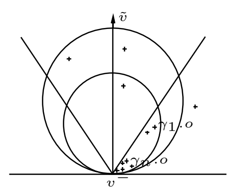

Figure 3. Lift of a right-horocyclic vector

Of course if v is a right horocyclic vector, then v^- is horospherical, and equivalently v is a horospherical vector in the sense of [**C**].

Remark 3.6. This definition depends only of v^- (indeed, if w is another vector with $v^- = w^-$, any cone around w is included in a cone around v). A point $\xi \in \Lambda_\Gamma$ which is the negative endpoint of a right horocyclic vector will therefore be called a *right horocyclic point*.

Lemma 3.7. *Let S be a non elementary hyperbolic surface. A vector $v \in T^1 S$ is a right horocyclic vector if and only if there exists $z \in \Omega$ such that for all α, there exists a sequence $t_n \to +\infty$, $v_n \in W^{su}_+(v)$ s.t. $g^{-t_n} v_n$ converges to $z \in \Omega$, but $g^{-t_n}\tilde{v}_n \notin \mathcal{C}(\tilde{v}, \alpha)$, where $\tilde{v}$ and $\tilde{v_n}$ are lifts resp. of v and v_n on the same horocycle of $T^1\mathbb{D}$.*

The definition of right horocyclic vector is easier, but the above equivalent property will be more useful in the sequel.

Proof. Let us begin with the following elementary fact.

Fact 3.8. *There exists $R > 0$, such that for all $\xi \in \Lambda_\Gamma$, there exists $\eta \in \Lambda_\Gamma$, such that the geodesic $(\xi\eta)$ intersects the ball $B(o, R)$.*

Indeed, assuming it is false, we could find a sequence $R_n \to \infty$, $\xi_n \in \Lambda_\Gamma$, $\xi_n \to \xi \in \Lambda_\Gamma$, s.t. for all $\eta \in \Lambda_\Gamma$, the distance $d(o,(\xi_n\eta))$ is greater than R_n. Passing to the limit, for $\eta \neq \xi$, we obtain $d(o,(\xi\eta)) = +\infty$, which gives a contradiction.

Now, let v be a right horocyclic vector. Let $D_n \to +\infty$, $\alpha_n \to +\infty$, and $\tilde{v}$ be a lift of v to $T^1\mathbb{D}$. There exists a point $\gamma_n.o$ in $Hor^+(g^{-D_n}\tilde{v}) \setminus \mathcal{C}(\tilde{v},\alpha)$. Using fact 3.8, we can find $\eta \in \Lambda_\Gamma$, $\eta \neq v^-$, s.t. the geodesic $(\tilde{v}^-\eta)$ intersects the ball $B(\gamma_n.o,R)$. Choose a vector $\tilde{w}_n \in \widetilde{\Omega} \cap T^1B(\gamma_n.o,R)$ tangent to this geodesic. It satisfies $w_n^- = v^-$, $w_n^+ = \eta$, $\tilde{w}_n = g^{-t_n}\tilde{v}_n$, $t_n \geq D_n - R$, $\tilde{v}_n \in W^{su}_+(\tilde{v})$, and $\tilde{w}_n$ does not belong to the cone $\mathcal{C}(\tilde{v},\alpha_n - R)$. Passing to T^1S, we get a sequence of vectors w_n of the compact set $T^1B(o,R)\cap\Omega$. Up to a subsequence, it converges to some $z \in \Omega$. We proved that there exists $z \in \Omega$, s.t. for all $\alpha > 0$, there exists $t_n \to +\infty$, and $v_n \in W^{su}_+(v)$ s.t. $g^{-t_n}v_n \to z$, and $g^{-t_n}\tilde{v}_n \notin \mathcal{C}(\tilde{v},\alpha)$.

Conversely, assume the existence of such a $z \in \Omega$. Fix $\alpha > 0$ and $D > 0$. Let $\rho = d(o,\pi(z))$, $\alpha > 0$, and $\beta = \alpha + \rho + 1$. There exists $t_n \to \infty$, $v_n \in W^{su}_+(v)$, $g^{-t_n}v_n \to z$, and $\tilde{v}_n \notin \mathcal{C}(\tilde{v},\beta)$. For n large enough, $t_n \geq D + \rho + 1$, and $d(g^{-t_n}v_n, z) \leq 1$. There exists an element $\gamma_n.o \in B(\pi(g^{-t_n}\tilde{v}_n),\rho)$. By construction, this element is in $Hor^+(g^{-D}\tilde{v})\setminus\mathcal{C}(g^{-D}\tilde{v},\alpha)$. Thus, $\tilde{v}$ is a right horocyclic vector. ☐

Proof of theorem 1.1. We will prove

Proposition 3.9. *Let S be a nonelementary hyperbolic surface. A vector $\tilde{v} \in T^1\mathbb{D}$ is a right horocyclic vector if and only if $W^{su}_+(v)$ is dense in Ω, where $v \in T^1S$ is the projection of $\tilde{v}$.*

and

Lemma 3.10. *Let S be a nonelementary geometrically finite surface. If $v^- \in \Lambda_{\mathrm{hor}}$, v is a right horocyclic point iff v^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$.*

Theorem 1.1 is an immediate consequence of these two results. Let us now prove them.

Fact 3.11. *If $y \in \overline{W^{su}_+(x)}$, then $W^{su}_+(y) \subset \overline{W^{su}_+(x)}$.*

Proof. Evident with the parametrization of W^{su} by the horocyclic flow. ☐

For a vector $v \in T^1S$, we denote by $\tilde{v}$ a lift to $T^1\mathbb{D}$, and by $v^\pm \in S^1$ the enpoints of this lift on the boundary.

Proposition 3.12. *Let S be a nonelementary surface. If $p \in \Omega$ is a periodic vector for the geodesic flow, then $W^{su}_+(p)$ is dense in Ω if and only if p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$.*

This result is valid on any nonelementary negatively curved surface, without geometrical finiteness assumption.

Recall first that on a nonelementary negatively curved surface, the *length spectrum* is non arithmetic (see [**Da**]), that is the set of lengths of periodic orbits $\{l(\gamma), <\gamma> \text{ periodic}\}$ generates a dense subgroup of $\mathbb{R}$.

Proof. Note first that if $p \in T^1S$ is a periodic vector for the geodesic flow and p^- is the first endpoint of an interval $]p^-\eta[$ of $S^1 \setminus \Lambda_\Gamma$, then $W^{su}_+(p)$ cannot be dense in Ω. Indeed, the vectors of $W^{su}_+(p)$ pointing in $]p^-\eta[$ do not even belong to Ω.

Assume now that p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$. We follow [**C**] almost verbatim.

First, $g^{\mathbb{R}} W^{su}_+(p)$ is dense in Ω (see also [**C**, Lemma1]). Indeed, there exists $x \in \Omega$, s.t. $(g^t x)_{t \geq 0}$ is dense in Ω. Let $\tilde{x}$ (resp. $\tilde{p}$) be a lift of x (resp. p) to $T^1\mathbb{D}$, and x^+ its positive endpoint in S^1. The orbit $\Gamma.x^+$ is dense in Λ_Γ. As p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, we can find a sequence $x_n^+ \in \Gamma.x^+$ converging to p^-, with $x_n^+ \geq p^-$ (in the counterclockwise order).

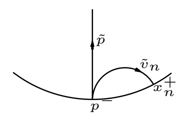

Figure 4. Construction of a dense geodesic in the weak unstable manifold $W^{wu}(p)$

If n is large enough, the unique vector $\tilde{v}_n$ of $W^{ss}(\tilde{p}) \cap (p^- x_n^+)$ belongs to the positive half-horocycle $W^{su}_+(\tilde{p})$, so that on T^1S, $v_n \in W^{su}_+(p)$ and $g^{\mathbb{R}} v_n$ is dense in Ω. Therefore, $g^{\mathbb{R}} W^{su}_+(p)$ is dense in Ω.

Now, let us show that $W^{su}_+(p)$ is dense in $g^{\mathbb{R}} W^{su}_+(p)$. (We still follow [**C**]). Fix $\varepsilon > 0$, and a periodic vector $p_0 \in T^1S$, s.t. $\exists m, n \in \mathbb{Z}$, with $|ml(p) + nl(p_0)| < \varepsilon$. Without loss of generality, assume $n \geq 0$. Let $\delta = \delta(\varepsilon, p_0) > 0$ be the constant appearing in the local product structure property around p_0 (see end of section 2).

As p^- is not the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$, we can lift p_0 to $\tilde{p_0}$ in such a way that $p_0^+ \in [p^- p^+]$.

Let $v \in W^{su}_+(p) \cap W^{ws}(p_0)$ be the vector obtained as the projection on T^1S of the unique $\tilde{v}$ of $W^{su}_+(\tilde{p}) \cap W^{ws}(\tilde{p}_0)$. If p_0^+ is well chosen (i.e. close enough to p^-), we have $W^{su}_{2\varepsilon}(v) \subset W^{su}_+(p)$. The geodesic orbit $(g^t v)_{t \in \mathbb{R}}$ is negatively asymptotic to the periodic orbit of p, positively asymptotic to the periodic orbit of p_0.

The end of the proof consists in using once again the local product structure to construct an orbit which is negatively asymptotic to the orbit of p, positively asymptotic to the orbit of p, and "in the middle" goes from the orbit of p to the orbit of p_0 following the orbit of v, turns a certain number of times around the periodic orbit of p_0, and later comes back to the orbit of p.

Choosing the number of turns around the orbit of p_0 will allow to construct vectors of $W^{su}_+(p)$ arbitrarily close to any vector of the closed orbit $(g^t p)_{t \in \mathbb{R}}$.

First, the vector v belongs to $W^{su}_+(p)$, and p is periodic, so that there exists $\tau \geq 0$, satisfying $g^{-\tau} v \in W^{su}_\varepsilon(g^{-\tau} p) = W^{su}_\varepsilon(p)$. Choose the smallest such τ_1, and let $v_1 = g^{-\tau_1} v$. There exists $s_1 \geq \tau_1$ s.t. $g^{s_1} v_1 = g^{s_1 - \tau_1} v \in W^{ss}_{\delta/2}(p_0)$; s_1 is the "time" needed to come from an ϵ-unstable neighbourhood of p to a $\delta/2$-neighbourhood of the orbit of p_0, following the orbit of v. As p_0 is periodic, note that for all nonnegative integer $i \geq 0$, $g^{s_1 + il(p_0)} v_1 \in W^{ss}_{\delta/2}(p_0)$.

There exists a vector $w \in W^{su}_{\delta/2}(p_0) \cap W^{ws}(p)$. Let $s_2 > 0$ s.t. $g^{s_2} w \in W^{ss}_\varepsilon(p)$.

For all $k \in \mathbb{N}$, ass $d(w, g^{s_1 + k.n.l(p_0)} v_1) \leq \delta$, we use the local product structure of the geodesic flow, and obtain a vector of $W^{su}_\varepsilon(g^{s_1 + knl(p_0) \pm \varepsilon} v_1) \cap W^{ss}_\varepsilon(w)$. The resulting geodesic orbit on T^1S is negatively and positively asymptotic to the orbit

of p, going from p to p_0, ε-shadowing the orbit of p_0 during the time $k.nl(p_0)$ and coming back to the orbit of p.

The key point (and the only difference with [**C**]) is that the "gluing" was done between some vector $g^t v$, $t \geq 0$ of the *positive* geodesic orbit of v, and a vector "coming back" from p_0 to p. It ensures that the resulting geodesic orbit intersects $W^{su}_{\varepsilon}(g^t v)$. As $W^{su}_{\varepsilon}(v)$ contains $g^{-t}W^{su}_{\varepsilon}(g^t v)$, this orbit intersect therefore $W^{su}_{\varepsilon}(v)$ which is included in the positive unstable horocycle $W^{su}_{+}(p)$.

Note that the time s_1 needed to go from p to p_0, and the time s_2 to come back, depend only on ε, and not on k, so that we can choose $k \in \mathbb{N}$ as large as we need.

Let us repeat verbatim the final argument of [**C**]. For all $\varepsilon > 0$, we found $s_1 > 0$, $s_2 > 0$, s.t. for all $k \in \mathbb{N}$, there exists $u \in W^{ss}_{2\varepsilon}(p)$, and $s_k \in \mathbb{R}$, with $|s_k - s_1 - s_2| < 2\varepsilon$, and $g^{-s_k - knl(p_0)}u \in W^{su}_{\varepsilon}(p)$. Let $j \in \mathbb{Z}$ be the greatest integer such that $jl(p) < -s_k - knl(p_0)$. Then, $g^{jl(p)}u = g^{jl(p)+s_k+knl(p_0)}g^{-s_k-knl(p_0)}u \in W^{ss}(p)$ is ε-close to the vector $g^{s_k+kl(p_0)}p$ on the periodic orbit of p. This vector also coincides with $g^{s_k+knl(p_0)+m'l(p)}p$ for all $m' \in \mathbb{Z}$. In particular, taking $m' = km$, we find a vector on $W^{ss}(p)$ very close to $g^{s_1+s_2+k(ml(p_0)+nl(p))}p$ for all positive integers $k \in \mathbb{N}$. As the length spectrum is non arithmetic, any point on the (periodic) geodesic orbit of p is ε close to such a point. Thus, $\overline{W^{su}_{+}(p)}$ contains $(g^t p)_{t\in\mathbb{R}}$, and therefore also $g^{\mathbb{R}}W^{su}_{+}(p)$ which is dense in Ω. This ends the proof. $\square$

Proof of proposition 3.9 The case of periodic vectors p follows from proposition 3.12 and the proof of lemma 3.10. We consider now only nonperiodic vectors.

Assume first that $W^{su}_{+}(v)\cap\Omega$ is dense in Ω, and prove that v is a right horocyclic vector.

Let p be a vector on a periodic geodesic, $l(p)$ its length, and $d(p)$ the distance between o and its orbit. Fix $\alpha > 0$ and $D > 0$. Without loss of generality, we assume $D \geq l(p) + d(p) + 2$. Consider the cone $\mathcal{C} = \mathcal{C}(g^{-D}\tilde{v}, \alpha)$, where $\tilde{v}$ is a lift of v to $T^1\mathbb{D}$. Remark that the distance between (the basepoint of) $h^s(g^{-D}\tilde{v})$ and the cone $\mathcal{C}$ goes to infinity when $s \to +\infty$.

Fix $\varepsilon \in]0,1[$. By density of $W^{su}_{+}(v)$ in Ω, we can find an infinite sequence $v_k \in W^{su}_{+}(v)$, $v_k = h^{s_k}v$, $s_k \to \infty$, s.t. v_k is so-close to p that $(g^{-t}v_k)_{0\leq t\leq 2D}$ and $(g^{-t}p)_{0\leq t\leq 2D}$ stay ε-close each other. We deduce that $g^{-2D}v_k$ is at distance ε from $g^{-2D}p$, hence from the orbit of p, and therefore at distance less than $1 + l(p) + d(p)$ from the projection $\pi(o)$ of o on S. Lift v to $\tilde{v} \in T^1\mathbb{D}$, and v_k to $\tilde{v_k} \in W^{su}_{+}(\tilde{v})$ As $v_k = h^{s_k}v$ goes to infinity on $W^{su}_{+}(v)$, the distance between $g^{-2D}\tilde{v_k}$ and $\mathcal{C}$ goes to infinity. Therefore, we can assume this distance be greater than $l(p) + d(p) + 2$. There exists a point of $\Gamma.o$ at distance at most $d(p) + l(p) + 1$ of $g^{-2D}v_k$. By construction, this point is inside $Hor^{+}(g^{-D}v) \setminus \mathcal{C}(v, \alpha)$. This construction works for all $\alpha > 0$ and $D > 0$ large enough, so that $\tilde{v}$ is a right horospherical vector.

Let us establish now the other direction, adapting methods of [**C**]. Let v be a right horocyclic vector. We will prove that there exists a periodic vector $p \in \overline{W^{su}_{+}(v)}$, with $W^{su}_{+}(p)$ dense in Ω.

Let $t_n \to \infty$, $v_n \in W^{su}_{+}(v)\cap\Omega$, $v_n \to \infty$ on the leaf, s.t. $g^{-t_n}v_n$ converges to some $z \in \Omega$, with $g^{-t_n}\tilde{v}_n$ staying outside a given cone $\mathcal{C}(v, 2)$. Let p be a periodic vector s.t. $W^{su}_{+}(p)$ is dense in Ω. Choose $\varepsilon_k \to 0$ and let δ_k be the constant associated to ε_k by the local product structure property around z.

Using this product structure, we construct an orbit negatively asymptotic to the negative orbit of z, and positively asymptotic to the orbit of p. More precisely, we can find $s_k \geq 0$, and $w_k \in W^{su}_{\delta_k/2}(z) \cap W^{ws}(p)$, s.t. for all $t \geq s_k$, $g^t w_k$ is ε_k-close to the orbit of p. Note that w_k is $\delta_k/2$-close to z.

Now, let n_k be large enough so that $t_{n_k} \geq 2s_k$ and $d(g^{-t_{n_k}} v_{n_k}, z) \leq \delta_k/2$. In particular, the distance between $g^{-t_{n_k}} v_{n_k}$ and w_k is at most δ_k.

As $v_{n_k} \in W^{su}_+(v)$ and $g^{-t_{n_k}} v_{n_k}$ is not in the cone $\mathcal{C}(v,2)$, the local strong stable manifold $W^{su}_{2\varepsilon_k}(g^{-t_{n_k}} v_{n_k})$ is included in $W^{su}_+(g^{-t_{n_k}} v)$. This fact will be crucial for the end of the proof; indeed, we will now glue the past orbit of $g^{-t_{n_k}} v_{n_k}$ with the future orbit of w_k, and the resulting orbit intersects the *positive* horocyclic orbit $W^{su}_+(v)$. Let us detail this gluing. Let $\tilde{v}$ be a lift of v, $\tilde{v}_{n_k}$ the lift of v_{n_k} on $W^{su}_+(\tilde{v})$, $\tilde{z}_k$ (resp. $\tilde{w}_k$) be the lift of z (resp w_k) $\delta_k/2$-close to $g^{-t_{n_k}} \tilde{v}_{n_k}$. Consider the geodesic joining v^- to $\tilde{w}_k^+$. By the above, this geodesic crosses $W^{su}_{2\varepsilon_k}(g^{-t_{n_k}} v_{n_k}) \subset W^{su}_+(g^{-t_{n_k}} v)$, and therefore also $W^{su}_+(\tilde{v})$.

Let $\tilde{y}_k$ be the unique vector of $W^{su}_+(\tilde{v})$ on this geodesic. By construction $(g^{-t}\tilde{y}_k)_{t\geq 0}$ is asymptotic to v^-, and $g^{-t}\tilde{y}_k$ belongs to a $2\varepsilon_k$-neighbourhood of $\tilde{w}_k$ for $t \simeq t_{n_k}$, and then it becomes positively asymptotic to $(g^t \tilde{w}_k)_{t \geq -t_{n_k}}$. In particular, on $T^1\mathbb{D}$, as s_k is the "time" needed on the orbit of w_k to join the ε_k-neighbourhood of the orbit of p, for $t \geq s_k - t_{n_k}$, the orbit of y_k becomes $2\varepsilon_k$-close to the orbit of p. We chose $t_{n_k} \geq 2s_k$ so that for $t = 0$, y_k is $2\varepsilon_k$-close to the orbit of p.

As this orbit is a compact set, up to a subsequence, we can assume that y_k converges. It implies that there exists $0 \leq \sigma \leq l(p)$ st $g^\sigma p \in \overline{W^{su}_+(v)}$. Of course $g^\sigma p$ is periodic and $W^{su}_+(g^\sigma p)$ is dense in Ω. Fact 3.11 implies now that $W^{su}_+(v) \cap \Omega$ is dense in Ω.
□

Proof of lemma 3.10 Assume first that v^- is the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$. As the property of being right horospherical or not depends only on v^-, we can assume that v is a periodic vector on the closed geodesic closing the funnel.

By definition of a funnel, it becomes clear that if o was chosen in a lift of the compact part of S, the intersection of the open right horoball $Hor^+(v)$ with $\Gamma.o$ is empty. Thus, $\tilde{v}$ is not a right horospherical vector.

Suppose now that v is not a right horospherical vector. There exists a cone $\mathcal{C}(v,\alpha)$ and a right horoball $Hor^+(g^{-T}v)$, s.t. $\Gamma.o$ does not intersect $Hor^+(g^{-T}v) \setminus \mathcal{C}(v,\alpha)$. Let us shrink the horoball from a distance d equal to the diameter of the compact part $C(S)$ of S. Thus, the set $Hor^+(g^{-T-d}v) \setminus \mathcal{C}(v,\alpha)$ does not intersect the Γ-orbit $\Gamma.\widetilde{C(s)}$ of the lift of the compact part. In other words, viewed on S, the projection of $Hor^+(g^{-T-d}v) \setminus \mathcal{C}(v,\alpha)$, which is a connected set, is necessarily included in a cusp or a funnel. It implies immediately that v^- is a parabolic point or is the first endpoint of an interval of $S^1 \setminus \Lambda_\Gamma$. By assumption, v^- cannot be parabolic, so that the result is proven. □

Geometrically infinite surfaces. On these surfaces, the situation is -not surprisingly - more complicated, and we only discuss here partial results on the behaviour of positive (resp. negative) half-horocycles.

Proposition 3.12 gives a complete answer for periodic vectors. Recall the

Theorem 3.13 (Hedlund, [**H**], thm 4.2)**.** *Let $S = \mathbb{D}/\Gamma$ be a hyperbolic surface of the first kind, i.e. such that $\Lambda_\Gamma = S^1$. Let $v \in T^1S$ be s.t. $(g^{-t}v)_{t\geq 0}$ returns*

infinitely often in a compact set. Then the positive half-horocycle $(h^s v)_{s\geq 0}$ *is dense in* T^1S*.*

In [**Sa-Scha**], in the case of an abelian cover of a compact surface (also a surface of the first kind), we proved the equidistribution, and therefore the density of all positive half-horocyclic orbits $(h^s v)_{s\geq 0}$ of vectors v whose asymptotic cycle is not maximal.

Question 3.14. It would be interesting to understand completely the behaviour of half-horocycles. For example,

(1) On a surface of the first kind ($\Lambda_\Gamma = S^1$), are all horospherical vectors also right horocyclic vectors (generalization of Hedlund's theorem) ? Or can we find a counterexample ?
(2) On a surface of the second kind, can we construct counterexamples to Hedlund's theorem? Or sufficient conditions to be right horocyclic ?

4. Proof of Theorem 1.2

In this section, S is a nonelementary geometrically finite surface.

Measures. Let δ_Γ be the critical exponent of Γ, defined by $\delta_\Gamma := \limsup_{T\to\infty} \#\{\gamma \in \Gamma, d(o,\gamma.o) \leq T\}$. The well known Patterson construction provides a *conformal density* of exponent δ_Γ on S^1, that is a collection $(\nu_x)_{x\in\mathbb{D}}$ of measures, supported on $\Lambda_\Gamma \subset S^1$, s.t. $\nu_o(S^1) = 1$, $\gamma_*\nu_x = \nu_{\gamma.x}$ for all $\gamma \in \Gamma$, and $\frac{d\nu_x}{\delta\nu_y}(\xi) = \exp(-\delta_\Gamma\beta_\xi(x,y))$.

The Patterson-Sullivan measure m^{ps} on T^1S, or Bowen-Margulis measure, is defined locally as the product

$$dm^{ps}(v) = \exp\left(\delta_\Gamma\beta_{v^-}(o,\pi(v)) + \delta_\Gamma\beta_{v^+}(o,\pi(v))\right) d\nu_o(v^-)d\nu_o(v^+)dt$$

in the coordinates $\Omega \simeq \Gamma\backslash(\Lambda_\Gamma^2 \times \mathbb{R})$.

Under our assumptions on S, it is well known [**Su**] that the Bowen-Margulis measure is finite and ergodic[1], that there exists a unique conformal density of exponent δ_Γ, that all measures ν_x are nonatomic and give full measure to the radial limit set. Moreover, the Bowen-Margulis-Patterson-Sullivan measure is the measure of maximal entropy of the geodesic flow, and it is fully supported on the nonwandering set Ω of the geodesic flow.

Denote by $\mu^{ps}_{H^+}$ the conditional measure of m^{ps} on the strong unstable horocycle $H^+(u) = (h^s u)_{s\in\mathbb{R}}$. It satisfies $d\mu^{ps}_{H^+}(v) = \exp\left(\delta_\Gamma\beta_{v^+}(o,\pi(v))\right) d\nu_o(v^+)$. To the measure m^{ps} is also associated a *transverse measure* invariant by the horocyclic foliation, that is a collection (μ_T) of measures on all transversals T to the strong unstable foliation, invariant by all holonomies of the foliation.

The classification of ergodic invariant measures for the horocyclic flow is well known ([**Bu**], [**Ro**]). Except the probability measures supported on periodic horocycles, and the infinite measures supported on wandering horocycles, *there is a unique ergodic invariant measure fully supported in the nonwandering set* $\mathcal{E} \simeq \Gamma\backslash(\Lambda_\Gamma \times S^1 \times \mathbb{R})$ of $(h^s)_{s\in\mathbb{R}}$. It is an infinite measure, defined locally by

$$dm(v) = ds(v)\exp\left(\delta_\Gamma\beta_{v^-}(o,\pi(v))\right) d\nu_o(v^-)dt,$$

where $ds(v)$ denotes the natural Lebesgue measure on $(h^s v)_{s\in\mathbb{R}}$ associated with the parametrization by (h^s).

[1]In fact, this result is false in general in variable negative curvature, and the assumption $(*)$ added in section 5 ensures finiteness and ergodicity of this measure

Sketch of the proof. The strategy of the proof is exactly the same as in [**Scha1**] and [**Scha2**]. We consider 'one-sided versions' of all results of these articles. Due to the lengths of the proofs of technical results in [**Scha1**], we just recall the main arguments, and point out the few differences.

The main lines of the proof are as follows. We do not prove directly equidistribution of horocyclic orbits to the unique "interesting" ergodic invariant measure, because this measure is infinite. We consider auxiliary averages on horocycles. Using classical arguments (tightness in theorem 4.2 and classification of invariant measures due to Burger [**Bu**] and Roblin [**Ro**]), we prove equidistribution of these auxiliary averages towards the *finite* Bowen-Margulis measure (theorem 4.1). We deduce then theorem 1.2 from the preceding.

Let $\psi : T^1S \to \mathbb{R}$ be a continuous compactly supported map. Denote by $(h^s u)_a^b$ the segment of orbit $(h^s u)_{a\le s\le b}$. Consider the following averages:

$$M^+_{r,u}(\psi) = \frac{1}{\mu^{ps}_{H^+}((h^s u)_0^R)} \int_{(h^s u)_0^R} \psi(v)\, d\mu^{ps}_{H^+}(v)\,.$$

These averages are supported on Ω. We prove

Theorem 4.1. *Let S be a nonelementary geometrically finite hyperbolic surface, and $u \in \mathcal{E} \subset T^1S$. If the positive orbit $(h^s u)_{s\ge 0}$ is dense in Ω, then it is equidistributed: for all $\psi : T^1S \to \mathbb{R}$ continuous with compact support, we have*

$$M^+_{r,u}(\psi) \to \int_{T^1S} \psi\, dm^{ps}\,, \quad \textit{when} \quad r \to \infty.$$

As in [**Scha2**], we deduce easily theorem 1.2 of theorem 4.1. In the proof of theorem 4.1, the difficult parts are the classification of (h^s)-invariant measures (see [**Bu**] and [**Ro**]) and the following tightness argument.

Theorem 4.2. *Let S be a nonelementary geometrically finite hyperbolic surface, and $u \in \mathcal{E} \subset T^1S$. For all $\varepsilon > 0$, there exist a compact set $K_{\varepsilon,u} \subset \Omega$ and $r_0 > 0$ such that for $r \ge r_0$, $M^+_{r,u}(K_{\varepsilon,u}) \ge 1 - \varepsilon$.*

Proof of theorem 4.2. This is the only non immediate part. For simplicity, assume that S has exactly one cusp. (If it has no cusp, Ω is compact, so that theorem 4.2 is obvious.) Denote by $(\xi_i)_{i\in\mathbb{N}} = \Gamma.\xi_1$ the Γ-orbit of parabolic limit points of Λ_Γ. As S is geometrically finite, there is a Γ-invariant family of disjoint horoballs H_i of $\mathbb{D}$, based at ξ_i, such that $\sqcup_{i\in\mathbb{N}} H_i = \Gamma.H_1$, Γ acts cocompactly on $(\Lambda^2_\Gamma \times \mathbb{R}) \setminus \cup_i T^1 H_i$. Assume that H_1 is the closest horoball to the origin o, that the distance from o to ∂H_1 is bounded by the diameter of the compact part C_0 of S, and that the geodesic ray $[o\xi_1)$ does not intersect other horoballs H_i, $i \ne 1$. Let Π be the subgroup of Γ stabilizing H_1. Its critical exponent δ_Π is equal to $1/2$ on hyperbolic surfaces. Moreover, $1 > \delta_\Gamma > \delta_\Pi$ for a nonlattice not convexe-compact geometrically finite group. ([2])

Let $o \in \mathbb{D}$ be fixed outside all horoballs H_i, $\xi \in S^1$ and $t \ge 0$. Let $\xi(t)$ be the point of the geodesic ray $[o\xi)$ at distance t of o, and define the set $V(o,\xi,t)$ as the

[2]Indeed, [**Pe**] as the surface has infinite volume, it contains a funnel. Let $\xi \notin \Lambda_\Gamma$, and $p \notin \Gamma$ a parabolic isometry fixing ξ. Using the divergence of Γ and [**DOP**, Prop.2], we obtain $1 \ge \delta_{<p>*\Gamma} > \delta_\Gamma$. As all parabolic subgroups of Γ are divergent, the same proposition [**DOP**, Prop.2] gives $\delta_\Gamma > \delta_\Pi$ for all Π parabolic subgroups of Γ.

set of points $\eta \in S^1$ whose projection on $[o\xi)$ is at distance at least t of o. By abuse of notation, we call such sets *shadows*, because they are comparable to Sullivan's shadows (it is a classical fact, see for example [**Scha1**]). We denote by $V(o,\xi,t)^+$ (resp. $V(o,\xi,t)^-$) the *positive* (resp. negative) *half-shadow*, that is the subset of points of $V(o,\xi,t)$ that are greater (resp. less) than ξ in the counterclockwise order.

If H_i is a horoball based at ξ_i, denote by s_i the distance between o and ∂H_i, or in other words the instant when the geodesic ray $[o\xi_i)$ enters in H_i. Notation $a(t) = B^{\pm 1}$ means that $\frac{1}{B} \le a(t) \le B$ for all $t \ge 0$.

Following exactly [**Scha2**, prop. 3.4], we get :

Proposition 4.3. *Let S be a geometrically finite hyperbolic surface. There is a constant $B > 0$ such that for all $\xi_i \in \Lambda_p$ and all $t \ge s_i$, where $s_i = \beta_{\xi_i}(o, \partial H_i)$, we have*

$$\nu_o(V(o,\xi_i,t)^+) = B^{\pm 1} e^{-\delta_\Gamma t} e^{(1-\delta_\Gamma)(t-s_i)}\,.$$

We will need the following immediate refinement of the above statement. Let $s \ge 0$ be large enough so that $Be^{(1-2\delta_\Gamma)s} \le \frac{1}{2B}$. We have

$$\nu_o(V(o,\xi_i,t)^+ \setminus V(o,\xi_i,t+s)^+) = (2B)^{\pm 1} e^{-\delta_\Gamma t} e^{(1-\delta_\Gamma)(t-s_i)}\,. \tag{4.1}$$

In other words, 'most' of the measure is in the boundary of the shadows. Of course, the same result holds for $V(o,\xi_i,t)^- \setminus V(o,\xi_i,t+s)^-$. It also holds, with a different constant, for $V(o,\xi_i,t)^+ \setminus V(o,\xi_i,t+s)$.

Consider now a horocycle $(h^s u)_{s\in\mathbb{R}}$, with $u^- \in \Lambda_{rad}$, $u \notin H_i$ for all $i \in \mathbb{N}$. Denote by $v_i = h^{\sigma_i} u$, $i \in \mathbb{N}$ the unique vector of this horocycle such that $v_i^+ = \xi_i$, and h_i the height of v_i in the horoball H_i (i.e. the unique real number s.t. $g^{-h_i} v_i \in T^1 \partial H_i$). As noticed in [**Scha2**, lemma 4.3], there is a (small) constant $\alpha > 0$, s.t.

$$(h^s v_i)_{|s| \le e^{(h_i-\alpha)/2}} \subset (h^s u)_0^{+\infty} \cap T^1 H_i \subset (h^s v_i)_{|s| \le e^{h_i/2}}\,.$$

For all $i \in \mathbb{N}$, define the horoball $H_i^N \subset H_i$ as the horoball s.t. the distance between ∂H_i and ∂H_i^N is equal to N. Using the above proposition, we prove that

Lemma 4.4. *Let $u \in \mathcal{E}$, with $u^- \in \Lambda_{rad}$. There is a constant $C > 0$ such that*

$$\frac{\mu^{ps}_{H^+}((h^s v_i)_{|s| \le e^{(h_i-N)/2}})}{\mu^{ps}_{H^+}((h^s v_i)_{e^{(h_i-N)/2} \le s \le e^{h_i/2}})} \le Ce^{-(2\delta_\Gamma - 1)N} \to 0 \quad \textit{when} \quad N \to \infty$$

uniformly in $i \in \mathbb{N}$.

This lemma says that the 'time' (measured with the measure $\mu^{ps}_{H^+}$) spent by a horocycle in a horoball H_i^N is small compared to the 'time' needed to go from ∂H_i^N to ∂H_i^0.

Recall that as S is a geometrically finite hyperbolic surface, $\delta_\Gamma > \delta_\Pi = 1/2$.

Proof. We only sketch the proof, and refer to [**Scha1**]. Let C_0 be the compact part of S, and $\tilde{C}_0$ a connected lift to $\mathbb{D}$ containing o.

As $u^- \in \Lambda_{rad}$, for all $i \in \mathbb{N}$, there exists $T_i \ge h_i/2$ s.t. $g^{-T_i} v_i \in \Gamma\tilde{C}_0$. By definition of $\mu^{ps}_{H^+}$ we get

$$\frac{\mu^{ps}_{H^+}((h^s v_i)_{|s| \le e^{(h_i-N)/2}})}{\mu^{ps}_{H^+}((h^s v_i)_{e^{(h_i-N)} \le s \le e^{h_i/2}})} = \frac{\mu^{ps}_{H^+}((h^s g^{-T_i} v_i)_{|s| \le e^{(h_i-N)/2-T_i}})}{\mu^{ps}_{H^+}((h^s g^{-T_i} v_i)_{e^{(h_i-N)/2-T_i} \le s \le e^{h_i/2-T_i}})}\,.$$

As the distance from $\pi(g^{-T_i}v_i)$ to $\gamma.o$ is less than the diameter of the compact part C_0, up to uniform constants, the above quantity is uniformly close to

$$\frac{\nu_{\gamma.o}(V(\gamma.o,\xi_i,T_i-h_i/2+N/2))}{\nu_{\gamma.o}\left(V(\gamma.o,\xi_i,T_i-h_i/2)^+\setminus V(\gamma.o,\xi_i,T_i-h_i/2+N/2)^+\right)}.$$

for some $\gamma\in\Gamma$. Proposition 4.3 and estimate (4.1) give the desired control. $\square$

Following [**Scha1**], we can now prove theorem 4.2. Define $\widetilde{K_{\varepsilon,u}} := \Lambda^2_\Gamma\times\mathbb{R}\setminus \sqcup_{i\in\mathbb{N}}T^1H_i^N$, for $N=N(\varepsilon)$ large enough. Denote by $I_{u,r,N}$ the set of $i\in\mathbb{N}$ such that $(h^su)_0^R$ intersects the unit tangent bundle $T^1H_i^N$ to the shrinked horoball H_i^N at height N inside H_i. As $(h^su)_{s\in\mathbb{R}}$ is one-dimensional, and $u\in T^1C_0$, for all $j\in I_{u,r,N}$ except maybe one boundary term denoted by $i_0=i_0(r)$, we have $(h^su)_0^r\cap T^1H_i^N=(h^su)_{s\in\mathbb{R}}\cap T^1H_i^N$. We deduce

$$\begin{aligned}
M^+_{r,u}(\sqcup_{i\in\mathbb{N}}T^1H_i^N) &\le \frac{\mu^{ps}_{H^+}(\sqcup_{i\in I_{u,r,N}}(h^su)_0^r\cap T^1H_i^N)}{\mu^{ps}_{H^+}(\sqcup_{i\in I_{u,r,N}}(h^su)_0^r\cap T^1H_i^0)}\\
&\le \frac{\sum_{j\in I_{u,r,N}\,,\,j\neq i_0(r)}\mu_{H^+}((h^sv_i)_{|s|\le e^{(h_i-N)/2}})+\mu_{H^+}((h^sv_{i_0})_{|s|\le e^{(h_{i_0}-N)/2}})}{\sum_{j\in I_{u,r,N}\,,\,j\neq i_0(r)}\mu_{H^+}((h^sv_i)_{|s|\le e^{h_i/2}})+\mu_{H^+}((h^sv_{i_0})_{-e^{h_i/2}}^{-e^{(h_i-N)/2}})}
\end{aligned}$$

Lemma 4.4 allows to conclude that for N large enough, uniformly in $r\ge 0$, the measure $M^+_{r,u}(\Gamma.T^1H_1^N)$ is less than ε.

Comparing with [**Scha1**], the only difference is that here there is only one boundary term $i_0(r)$.

Proof of theorem 4.1. We follow [**Scha2**]. Thanks to theorem 4.2, all limit points of $(M^+_{r,u})_{r\ge0}$ are probability measures. As in [**Scha2**, lemma 3.6], we observe that such a limit gives measure zero to the set of periodic horocycles.

Moreover, it is not difficult to see [**Scha2**, lemma 3.5] that a limit point of the family $(M^+_{r,u})_{r\ge0}$ when $r\to\infty$ can be written as the product of a transverse invariant measure to the strong unstable foliation by the measure $\mu^{ps}_{H^+}$. The only fact to check is that $\mu^{ps}_{H^+}((h^su)_{0\le s\le R})\to+\infty$ as $R\to\infty$, and it can be done by the argument of [**Scha2**, Lemma 4.2], usin the fact that the positive half-horocycle $(h^su)_{s\ge0}$ is dense.

The uniqueness ([**Ro**]) of a transverse measure of full support in the nonwandering set $\mathcal{E}$ giving measure 0 to periodic horocycles allows to conclude the proof.

Proof of theorem 1.2. As $(h^su)_0^{+\infty}$ is dense, and therefore recurrent, the proof is exactly the same as in [**Scha2**]. The idea is to restrict the attention to a small flow box B, and to compare the transverse measures on a transversal T of B induced on one side by the averages $M^+_{r,u}(\psi)$ and on the other side by ratios $\frac{\int_0^r\psi\circ h^su\,ds}{\int_0^r\mathbf{1}_B\circ h^su\,ds}$.

5. Surfaces with variable negative curvature

Most results proved here extend to surfaces of variable negative curvature. More precisely, we assume that all sectional curvatures are pinched between two negative constants. Some definitions of the notions used here differ sightly, and we refer to the preliminary sections of [**Scha1**] or [**Scha2**] for details. The main difference is that there is no canonical parametrization of horocycles by a nice horocyclic flow, even if it is possible to define such a flow (see [**Mrc**]).

The motivated reader can check that the proof of theorem 1.1 an all results of section 3 extend verbatim to the situation of pinched negatively curved surfaces.

Concerning the equidistribution, we need to be more careful. We add an assumption, denoted by $(*)$ in [**Scha1**] and [**Scha2**], which allows to control the geometry of the cusps, and ensures in particular that the Bowen-Margulis measure is finite. With this restriction, theorem 1.2 is valid on pinched negatively curved geometrically finite surfaces.

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